DETECTION OF GRAVITATIONAL WAVES FROM INSPIRALING COMPACT BINARIES USING NON-RESTRICTED POST-NEWTONIAN APPROXIMATIONS

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The set up of matched filters for the detection of gravitational waves from in-spiraling compact binaries is usually carried out using the restricted post-Newtonian approximation: the filter phase is modelled including post-Newtonian corrections, whereas the amplitude is retained at the Newtonian order. Here we investigate the effects of the introduction of post-Newtonian corrections also to the amplitude and we discuss some of the implications for signal detection and parameter estimation.

1 Introduction

It has long been recognized that binary systems of compact objects are important sources of gravitational waves (GW) both for the ground based interferometric detectors currently under construction [1,2] and future space-based interferometers like LISA [3].

In this paper, we consider the in-spiral phase of the coalescence of binary systems in circular orbit using post-Newtonian (PN) approximations to general relativity [4].

The detection of in-spiral signals is carried out by cross-correlating the detector output with a discrete set of filters [5,6], usually computed within the so-called restricted post-Newtonian approximation [5]: PN corrections are taken into account in the phase of the wave-form, whereas the amplitude is retained at the lowest Newtonian order. Thus, one discards all multipole components except the quadrupole one. Such simplification of the filter structure is believed to have negligible effects on the detection performances and is not expected to affect appreciably the statistical errors in the estimation of the source parameters.

Here we investigate the effects of the introduction of PN corrections also to the amplitude of the wave-form and discuss some implications for signal detection and parameter estimation. For sake of simplicity we will assume negligible spins and, as it is always the case for ground based experiments, circular orbits.

2 The wave-form

The signal produced at the output of an interferometric detector by a gravitational wave of polarization amplitudes h_+ and $h_\times$ can be written as

$$h(t) = F_+ h_+(t) + F_\times h_\times(t) \ , \tag{1}$$

where F_+ and $F_\times$ are the so-called beam pattern functions of the detector [7]; they depend on the location of the source in the sky (θ,ϕ) and the polarization angle ψ. If we consider the in-spiral of a binary system of masses m_1 and m_2, h_+ and $h_\times$ read [8]

$$h_{+,\times} = \frac{2m\eta}{r} x \left\{ H^{(0)}_{+,\times} + x^{1/2} H^{(1/2)}_{+,\times} + x H^{(1)}_{+,\times} + x^{3/2} H^{(3/2)}_{+,\times} + x^2 H^{(2)}_{+,\times} + \cdots \right\} , \tag{2}$$

where $x \equiv (m\omega)^{2/3}$, ω is the system orbital frequency and r the source distance; $m = m_1 + m_2$, $\mu = m_1 m_2/m$, $\eta = \mu/m$ and $\mathcal{M} = \mu^{3/5} m^{2/5} = m\eta^{3/5}$ are the total mass, the reduced mass, the symmetric mass ratio and the chirp mass, respectively.

The lower terms of the PN expansion for the plus and cross polarization are given by [8]

$$\begin{aligned}
H^{(0)}_{+} &= -(1+c^2)\cos\Phi , && (3)\\
H^{(1/2)}_{+} &= -s\,\frac{\sqrt{1-4\eta}}{8}\left[(5+c^2)\cos(\frac{1}{2}\Phi) - 9(1+c^2)\cos(\frac{3}{2}\Phi)\right] , && (4)\\
H^{(0)}_{\times} &= -2c\,\sin\Phi , && (5)\\
H^{(1/2)}_{\times} &= -\frac{3}{4}\,s\,c\,\sqrt{1-4\eta}\left[\sin(\frac{1}{2}\Phi) - 3\sin(\frac{3}{2}\Phi)\right] , && (6)
\end{aligned}$$

where $c = \cos\iota$ and $s = \sin\iota$; ι is the angle between the direction of the source and the orbital angular momentum, and Φ is twice the orbital phase.

If one considers only the first term in (2), corresponding to the Newtonian one, one gets the restricted PN approximation. Our goal is to study the full 2 PN wave-form; here we will present some preliminary results, where the 0.5 PN corrections to the amplitude are taken into account, keeping however the phase at 2 PN order. Although this is a simplified and, to some extent, arbitrary choice of signal, all new features of the wave-form are introduced, in particular more information about the masses and the position of the source.

Considering the amplitude through 0.5-PN order, the GW output at the detector can be written as

$$h(t) = \frac{2m\eta}{r}(m\pi F)^{2/3}\left\{h^{(0)}(t) + (m\pi F)^{1/3} h^{(1/2)}(t)\right\} , \tag{7}$$

where F is the quadrupole gravitational wave frequency, i.e., $d\Phi/dt = 2\pi F$, and

$$h^{(0)}(t) = -\sqrt{F_+^2(1+c^2)^2 + F_\times^2 4c^2}\,\cos(\Phi + \varphi_{(0)}) , \tag{8}$$

$$\varphi_{(0)} = \arctan\left\{\frac{-2cF_\times}{(1+c^2)F_+}\right\} , \tag{9}$$

$$\begin{aligned}
h^{(1/2)}(t) = s\frac{\sqrt{1-4\eta}}{4} \quad & \left[-\sqrt{F_+^2\frac{(5+c^2)^2}{4} + F_\times^2 9c^2}\,\cos(\frac{1}{2}\Phi + \varphi_{(1/2)})\right. \\
& \left. + \frac{9}{2}\sqrt{F_+^2(1+c^2)^2 + F_\times^2 4c^2}\,\cos(\frac{3}{2}\Phi + \varphi_{(0)})\right] ,
\end{aligned} \tag{10}$$

$$\varphi_{(1/2)} = \arctan\left\{\frac{-6cF_\times}{(5+c^2)F_+}\right\} . \tag{11}$$

The Fourier transform of $h(t)$, calculated using the stationary phase approximation [5,9], reads:

$$\tilde{h}(\nu) = -\sqrt{F_+^2(1+c^2)^2 + F_\times^2 4c^2}\sqrt{\frac{5\pi}{96}}\frac{(\pi\nu)^{-7/6}}{r}\mathcal{M}^{5/6}\exp\left[i\left(2\pi\nu t_c + \Xi(\nu) - \varphi_{(0)} - \frac{\pi}{4}\right)\right]\Lambda \tag{12}$$

where

$$\begin{aligned}\Lambda = \quad & 1+\frac{s}{4}\sqrt{1-4\eta}\sqrt{\frac{F_+^2(5+c^2)^2/4+F_\times^2 9c^2}{F_+^2(1+c^2)^2+F_\times^2 4c^2}}(\pi m\nu)^{1/3}\left(\frac{1}{2}\right)^{1/3}\times \\ & \exp\left[i\left(\frac{1}{2}\Xi(2\nu)-\Xi(\nu)+\varphi_{(0)}-\varphi_{(1/2)}\right)\right] \\ & -s\frac{9}{8}\sqrt{1-4\eta}\,(\pi m\nu)^{1/3}\left(\frac{3}{2}\right)^{1/3}\exp\left[i\left(\frac{3}{2}\Xi\left(\frac{2}{3}\nu\right)-\Xi(\nu)\right)\right]\ , \end{aligned} \tag{13}$$

and

$$\begin{aligned}\Xi(\nu) = -\phi_c+\frac{3}{4}(8\pi\mathcal{M}\nu)^{-5/3} \quad & \left[1+\frac{20}{9}\left(\frac{743}{336}+\frac{11}{4}\eta\right)(\pi m\nu)^{2/3}-16\pi(\pi m\nu)\right. \\ & \left.+10\left(\frac{3058673}{1016064}+\frac{5429}{1008}\eta+\frac{617}{144}\eta^2\right)(\pi m\nu)^{4/3}\right]\ . \end{aligned} \tag{14}$$

In the restricted PN approximation $\Lambda = 1$. Now it contains two additional contributions related to the 0.5 PN corrections to the amplitude. Notice the dependency of these two terms on $\sin\iota$ and $\sqrt{1-4\eta}$: the departure from the value $\Lambda = 1$ increases as $\iota \to \pi/2$ and $\eta \to 0$.

3 Formalism

We denote by h the "true" GW signal and $u(\boldsymbol{\theta})$ the family of templates, as a function of the parameter vector $\boldsymbol{\theta} = (t_c, \phi_c, \boldsymbol{\lambda})$. The signal to noise ratio SNR, for optimal filtering, is defined as [9]

$$\mathrm{SNR} = \sqrt{(h|h)}\ , \tag{15}$$

where $(\,|\,)$ denotes the usual inner product. The fraction of SNR obtained by cross-correlating a template $u(\boldsymbol{\theta})$ with h is given by the ambiguity function

$$\mathcal{A}(\boldsymbol{\theta}) = \frac{(h|u(\boldsymbol{\theta}))}{\sqrt{(h|h)\ (u(\boldsymbol{\theta})|u(\boldsymbol{\theta}))}}\ , \tag{16}$$

which depends on the choice of $\boldsymbol{\theta}$. The maximum of the ambiguity function over the whole parameter space is defined as the fitting factor [10]

$$FF = \max_{\boldsymbol{\theta}} \frac{(h|u(\boldsymbol{\theta}))}{\sqrt{(h|h)\ (u(\boldsymbol{\theta})|u(\boldsymbol{\theta}))}}\ . \tag{17}$$

The fitting factor is a measure of how well any chosen family of templates fits the signal h. The maximization of the ambiguity function over the extrinsic parameters ϕ_c and t_c, phase and time of coalescence, is the so-called match

$$M(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2) = \max_{\phi_c,t_c} \frac{(h(\boldsymbol{\lambda}_1)|u(\boldsymbol{\lambda}_2))\ e^{i(2\pi f t_c-\phi_c)}}{\sqrt{(h(\boldsymbol{\lambda}_1)|h(\boldsymbol{\lambda}_1))\ (u(\boldsymbol{\lambda}_2)|u(\boldsymbol{\lambda}_2))}}\ . \tag{18}$$

For the set up of the bank of filters, one sets a minimal match [6,11] as the match between signal and template in the case where the signal is equidistant from all the nearest templates.

4 Results

In the results presented in the following, the signal h is computed according to Eq. (13)-(15), whereas the template wave-forms are calculated within the usual restricted PN approximation,

Table 1: Comparison of the values of the fitting factors FF, and the ratios $\mathcal{R} = \sqrt{(u|u)/(h|h)}$ for the pair of masses 0.1-10 $M_\odot$, and 1.4-10 $M_\odot$, for different orientations in the sky and polarization angles.

Orientation-polarization	$\iota = \widehat{\mathbf{LN}}$	0.1-10 $M_\odot$ FF	0.1-10 $M_\odot$ $\mathcal{R}$	1.4-10 $M_\odot$ FF	1.4-10 $M_\odot$ $\mathcal{R}$
$\forall\, \theta,\, \phi,\, \psi$	$\pi/2$	0.9319	0.9319	0.9549	0.9560
$\theta = \pi/6,\ \phi = \pi/4,\ \psi = \pi/4$	$\pi/3$	0.9495	0.9495	0.9669	0.9677
$\theta = \pi/6,\ \phi = \pi/4,\ \psi = \pi/6$	$\pi/3$	0.9498	0.9498	0.9671	0.9681
$\theta = \pi/6,\ \phi = \pi/4,\ \psi = 0$	$\pi/3$	0.9516	0.9516	0.9683	0.9691
$\theta = \pi/6,\ \phi = \pi/4,\ \psi = \pi/4$	$\pi/4$	0.9662	0.9662	0.9780	0.9787
$\theta = 0,\ \phi = 0,\ \psi = 0$	$\pi/4$	0.9662	0.9662	0.9780	0.9787
$\theta = \pi/6,\ \phi = \pi/4,\ \psi = \pi/6$	$\pi/4$	0.9664	0.9664	0.9782	0.9789
$\theta = \pi/6,\ \phi = \pi/4,\ \psi = \pi/4$	$\pi/6$	0.9829	0.9829	0.9890	0.9895
$\theta = 0,\ \phi = 0,\ \psi = 0$	$\pi/6$	0.9829	0.9829	0.9890	0.9895
$\theta = \pi/6,\ \phi = \pi/4,\ \psi = \pi/6$	$\pi/6$	0.9829	0.9829	0.9890	0.9895
$\theta = \pi/6,\ \phi = \pi/4,\ \psi = 0$	$\pi/6$	0.9831	0.9831	0.9891	0.9896

corresponding to $\Lambda = 1$ in Eq. (14). The noise curve is the one corresponding to the initial LIGO configuration [11].

In table 1, we give the fitting factors and the ratios of SNR, $\mathcal{R} = \sqrt{(u|u)/(h|h)}$, for the pair of masses 0.1-10 $M_\odot$ and 1.4-10 $M_\odot$, for different orientation and polarization angles. We note that the fitting factor reaches a minimum for $\iota = \pi/2$ as expected, and monotonically increases as $\iota \to 0$ or $\iota \to \pi$; it depends rather weakly on θ, ϕ and ψ.

For a fix position in the sky and $\iota = \pi/2$, we calculate then the fitting factors for different mass pairs. The fitting factor varies from 0.87 to 1.0; FF gets smaller as m increases and/or η decreases, see table 2.

It is now interesting to investigate the loss of SNR if one uses a restricted PN bank of filters to detect signals that include PN amplitude terms. The discrete mesh of filters is normally generated in such a way that any signal in the restricted PN plane produces a match $M(u(\boldsymbol{\lambda}_1), u(\boldsymbol{\lambda}_2))$ is always larger than a minimum value, usually set to 0.97. Assuming that the "true" GW signal, h, is includes PN corrections to the amplitude, so that it lies outside the template space, we want to quantify the match $M(h(\boldsymbol{\lambda}_1), u(\boldsymbol{\lambda}_2))$ between the PN signal and the nearest restricted PN template.

In table 3, we present the results obtained for a system of masses 0.1-10$M_\odot$ and different

Table 2: Fitting factors and the ratios of SNR for $\iota = \pi/2$.

$m_1/M_\odot$	$m_2/M_\odot$	FF	$\mathcal{R}$	$m_1/M_\odot$	$m_2/M_\odot$	FF	$\mathcal{R}$
50	1.4	0.9012	0.8864	25	1.4	0.9139	0.9184
20	5	0.9602	0.9373	15	1.4	0.9360	0.9362
10	9	0.9997	0.9995	10	8	0.9987	0.9984
10	5	0.9892	0.9880	10	1.4	0.9549	0.9560
10	1.0	0.9487	0.9493	10	0.75	0.9446	0.9438
10	0.5	0.9399	0.9398	10	0.25	0.9350	0.9352
10	0.1	0.9319	0.9319	1.4	1.0	0.9991	0.9991
1.4	0.75	0.9974	0.9974	1.4	0.5	0.9941	0.9941
1.4	0.25	0.9885	0.9885	1.4	0.1	0.9834	0.9834

Table 3: Fitting factors and match when the GW signal and the template have different parameter values. Results for 0.1-10$M_\odot$.

ι	FF	$M(u(\boldsymbol{\lambda}_1), u(\boldsymbol{\lambda}_2))$	$M(h(\boldsymbol{\lambda}_1), u(\boldsymbol{\lambda}_2))$	$FF \times M(u(\boldsymbol{\lambda}_1), u(\boldsymbol{\lambda}_2))$
$\pi/2$	0.9319	0.9319	0.8684	0.8684
		0.9664	0.9006	0.9006
		0.9693	0.9032	0.9033
		0.9779	0.9113	0.9113
		0.9826	0.9157	0.9157
$\pi/3$	0.9495	0.9319	0.8847	0.8848
		0.9664	0.9176	0.9176
		0.9693	0.9203	0.9203
		0.9779	0.9285	0.9285
		0.9826	0.9329	0.9330
$\pi/4$	0.9662	0.9319	0.9003	0.9003
		0.9664	0.9337	0.9338
		0.9693	0.9365	0.9365
		0.9779	0.9448	0.9448
		0.9826	0.9494	0.9494
$\pi/6$	0.9829	0.9319	0.9159	0.9159
		0.9664	0.9499	0.9499
		0.9693	0.9527	0.9527
		0.9779	0.9612	0.9612
		0.9826	0.9658	0.9658

Table 4: Measurement errors at SNR=10.

	$\Delta A/A$	Δt_c	$\Delta\phi_c$	$\Delta F_\times/F_+$	$\Delta\cos\iota$	$\Delta\mathcal{M}/\mathcal{M}$	$\Delta\eta/\eta$
Restricted PN	0.1000	3.3281×10^{-3}	8.0357	-	-	8.4256×10^{-3}	0.1107
0.5 PN - 2 PN	0.1062	2.1179×10^{-3}	4.7571	0.9435	0.4291	4.2884×10^{-3}	0.0624

choices of ι. For each case we calculate the fitting factor between the signal and the family of templates; then, using different parameter values $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$, we calculate the match in the restricted PN plane $M(u(\boldsymbol{\lambda}_1), u(\boldsymbol{\lambda}_2))$, and, for the same parameters, the match between the signal and the template $M(h(\boldsymbol{\lambda}_1), u(\boldsymbol{\lambda}_2))$.

What we observe is that the "real match" can be approximated as $M(h(\boldsymbol{\lambda}_1), u(\boldsymbol{\lambda}_2)) \approx FF \times M(u(\boldsymbol{\lambda}_1), u(\boldsymbol{\lambda}_2))$.

We turn now attention to the issue of estimating the source parameters. It is important to notice that for a waveform computed taking into account PN corrections to the amplitude, two additional parameters are involved; our choice corresponds to $F_\times/F_+$ and $\cos\iota$. We adopt here the standard variance-covariance matrix analysis [9,12], although it can underestimate the statistical errors in the limit of low SNR [13,14]; such well-known problem is beyond the purposes of this work and for sake of simplicity we will remain into the usual frame of the computation of the Fisher information matrix.

In table 4, we compare the errors for SNR=10 between the restricted and the "non-restricted" PN approximation. The results refer to a system of 1.4-20$M_\odot$ and $\theta = \pi/6$, $\phi = \pi/4$, $\psi = \pi/4$, and $\iota = \pi/3$. The use of more accurate wave-forms leads to smaller errors (by roughly a factor $\simeq 2$) in the determination of the masses; however the information about ι and $F_\times/F_+$ remain

Table 5: Correlation coefficients. (a) Restricted 2 PN, (b) 0.5-2PN.

(a)	$\ln A$	t_c	ϕ_c	$\ln \mathcal{M}$	$\ln \eta$
$\ln A$	1.00000	0.00000	0.00000	0.00000	0.00000
t_c	0.00000	1.00000	0.99202	0.92966	0.97681
ϕ_c	-0.00000	0.99202	1.00000	0.96540	0.99576
$\ln \mathcal{M}$	0.00000	0.92966	0.96540	1.00000	0.98411
$\ln \eta$	-0.00000	0.97681	0.99576	0.98411	1.00000

(b)	$\ln A$	t_c	ϕ_c	$F_\times/F_+$	$\cos\iota$	$\ln \mathcal{M}$	$\ln \eta$
$\ln A$	1.00000	0.04913	0.04649	-0.00075	0.33173	0.03815	0.04394
t_c	0.04913	1.00000	0.96733	-0.13841	-0.00999	0.87264	0.95706
ϕ_c	0.04649	0.96733	1.00000	0.05476	-0.01925	0.92878	0.98488
$F_\times/F+$	-0.00075	-0.13841	0.05476	1.00000	-0.00209	-0.06104	-0.06759
$\cos\iota$	0.33173	-0.00999	-0.01925	-0.00209	1.00000	-0.02954	-0.02390
$\ln \mathcal{M}$	0.03815	0.87264	0.92878	-0.06104	-0.02954	1.00000	0.96912
$\ln \eta$	0.04394	0.95706	0.98488	-0.06759	-0.02390	0.96912	1.00000

very poor.

In table 5 we provide the correlation coefficients for the same parameter choice. We notice that the correlation coefficients are also smaller and that the two new parameters have small correlations with the other ones. Notice that the amplitude is now correlated with the other parameters while it is not the case for the restricted PN case.

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